Addendum: A Classification of Plane Symmetric Kinematic Self-similar Solutions

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In our recent paper, we classified plane symmetric kinematic self-similar perfect fluid and dust solutions of the second, zeroth and infinite kinds. However, we have missed some solutions during the process. In this short communication, we add up those missing solutions. We have found a total of seven solutions, out of which five turn out to be independent and cannot be found in the earlier paper.

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Recently, we presented a classification of kinematic self-similar plane symmetric spacetimes [1]. We have discussed the plane symmetric solutions that admit kinematic self-similar vectors of the second, zeroth, and infinite kinds when the perfect fluid is tilted to the fluid flow, parallel or orthogonal. However, we missed some cases that could provide more solutions. In this addendum, we present those missing solutions, which turn out to be five in number. Further, for the the self-similarity of the first kind (tilted), the two-fluid formalism does not work as the self-similar variable is $\xi = \frac{x}{t}$. We shall investigate a different approach to obtain the solution in this case. The tilted perfect fluid yields four more solutions (one first-kind solution, two 2nd-kind solutions and one zeroth-kind solution), the parallel perfect fluid gives one infinite kind solution, and the orthogonal perfect fluid provides two

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solutions (one first-kind solution and one 2nd-kind solution). Thus, we obtain total seven solutions out of which five solutions are independent. The plane symmetric metric considered in the paper [1] is the following:

$$ds^2 = e^{2\nu(t,x)}dt^2 - dx^2 - e^{2\lambda(t,x)}(dy^2 + dz^2). \tag{1}$$

We are skipping the details as the procedure can be seen elsewhere [1].

The tilted perfect fluid of the first kind implies that the energy density ρ and pressure p must take the following forms:

$$\begin{aligned} \kappa\rho &= \frac{1}{x^2}\rho(\xi), & (2)\\ \kappa p &= \frac{1}{x^2}p(\xi), & (3) \end{aligned}$$

where the self-similar variable is $\xi = x/t$. When the Einstein field equations (EFEs) and the equations of motion for the matter field are satisfied, a set of ordinary differential equations (ODEs) is obtained, hence, the EFEs and equations of motion [1] reduce to

$$\begin{aligned} \dot{\rho} &= -2\dot{\lambda}(\rho + p), & (4)\\ 2p - \dot{p} &= \dot{\nu}(\rho + p), & (5)\\ \rho &= -4\dot{\lambda} - 3\dot{\lambda}^2 - 2\ddot{\lambda} - 1, & (6)\\ 0 &= \dot{\lambda}^2, & (7)\\ 0 &= \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\lambda}\dot{\nu}, & (8)\\ p &= 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\nu} + 2\dot{\lambda}\dot{\nu}, & (9)\\ 0 &= 2\dot{\lambda}\dot{\nu} - 2\ddot{\lambda} - 3\dot{\lambda}^2 - 2\dot{\lambda}, & (10)\\ p &= \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda}\dot{\nu} + \ddot{\nu} + \dot{\nu}^2, & (11)\\ 0 &= -\ddot{\lambda} - \dot{\lambda}^2 - \dot{\lambda} + \dot{\lambda}\dot{\nu}. & (12) \end{aligned}$$

Only the EOS(3) is compatible with this kind. Equations (2) and (3) yield $p = k\rho$ while Eqs. (7) and (4) imply that λ and ρ, respectively, are arbitrary constants. Also, Eq. (5) gives $\dot{\nu} = \frac{2k}{k+1}$. Using this value in the remaining equations, we obtain the following solution:

$$\begin{aligned} \nu &= \ln\left(c_0\xi^{(1\mp\sqrt{2})}\right), \quad \lambda = c_2, \\ \rho &= constant, \quad k = -3 \pm \sqrt{2}. & (13) \end{aligned}$$

The corresponding metric is

$$ds^2 = (\frac{x}{t})^{(2\mp 2\sqrt{2})}dt^2 - dx^2 - x^2(dy^2 + dz^2). \tag{14}$$

For the self-similarity of the second kind, we obtain solutions only with the EOS(3), and these solutions are missing in Ref. 1. The energy density ρ and pressure p can be written as

$$\begin{aligned} \kappa\rho &= \frac{1}{x^2}[\rho_1(\xi) + \frac{x^2}{t^2}\rho_2(\xi)], && (15)\\ \kappa p &= \frac{1}{x^2}[p_1(\xi) + \frac{x^2}{t^2}p_2(\xi)], && (16) \end{aligned}$$

where the self-similar variable is $\xi = x/(\alpha t)^{\frac{1}{\alpha}}$. A set of ODEs yield

$$\begin{aligned} \dot{\rho_1} &= -2\dot{\lambda}(\rho_1 + p_1), && (17)\\ \dot{\rho_2} + 2\alpha\rho_2 &= -2\dot{\lambda}(\rho_2 + p_2), && (18)\\ -\dot{p_1} + 2p_1 &= \dot{\nu}(\rho_1 + p_1), && (19)\\ -\dot{p_2} &= \dot{\nu}(\rho_2 + p_2), && (20)\\ \rho_1 &= -4\dot{\lambda} - 3\dot{\lambda}^2 - 2\ddot{\lambda} - 1, && (21)\\ \alpha^2 e^{2\nu}\rho_2 &= \dot{\lambda}^2, && (22)\\ 0 &= \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\lambda}\dot{\nu}, && (23)\\ p_1 &= 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\nu} + 2\dot{\lambda}\dot{\nu}, && (24)\\ \alpha^2 e^{2\nu}p_2 &= -2\ddot{\lambda} - 3\dot{\lambda}^2 - 2\alpha\dot{\lambda} + 2\dot{\lambda}\dot{\nu}, && (25)\\ p_1 &= \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda}\dot{\nu} + \ddot{\nu} + \dot{\nu}^2, && (26)\\ \alpha^2 e^{2\nu}p_2 &= -\ddot{\lambda} - \dot{\lambda}^2 - \alpha\dot{\lambda} + \dot{\lambda}\dot{\nu}. && (27) \end{aligned}$$

Proceeding along the same lines with the EOS(3), as given in Ref. 1, for $k \neq -1$ we assume that $\rho_1 = 0$ and that ρ_2 is arbitrary. Thus, Eqs. (21), (23), and (24) show that $\dot{\nu} = 0$ and $\dot{\lambda} = -1$, and Eq. (18) implies that $\alpha = k + 1$. Equations (25) and (27) give $\alpha = 2$, and we obtain the following spacetime:

$$\begin{aligned} \nu &= c_1, \quad \lambda = -\ln\xi + c_2, \quad \rho_1 = 0 = p_1, \quad \rho_2 = constant = p_2,\\ \alpha &= 2, \quad k = 1. && (28) \end{aligned}$$

The resulting plane symmetric metric becomes

$$ds^2 = dt^2 - dx^2 - 2t(dy^2 + dz^2). \tag{29}$$

When $k \neq -1$, we take $\rho_2 = 0$, and ρ_1 is arbitrary; hence, Eq. (22) implies that $\dot{\lambda} = 0$. For the first possibility, it follows that

$$\begin{aligned} \nu &= \frac{2k}{k+1}\ln\xi + c_1, \quad \lambda = c_2, \quad p_1 = k\rho_1, \quad \rho_1 = constant, \\ p_2 &= 0 = \rho_2, \quad k = -3 \pm 2\sqrt{2}; \end{aligned} \tag{30}$$

hence, the plane symmetric spacetime will take the following form:

$$ds^2 = (\frac{x}{(\alpha t)^{1/\alpha}})^{\frac{4k}{k+1}} dt^2 - dx^2 - x^2(dy^2 + dz^2). \tag{31}$$

For the self-similarity of the zeroth kind, the EFEs show that the quantities ρ and p must be of the form

$$\kappa\rho = \frac{1}{x^2}[\rho_1(\xi) + x^2\rho_2(\xi)], \tag{32}$$

$$\kappa p = \frac{1}{x^2}[p_1(\xi) + x^2 p_2(\xi)], \tag{33}$$

where the self-similar variable is $\xi = \frac{x}{e^t}$. A set of ODEs follows such that

$$\dot{\rho_1} = -2\dot{\lambda}(\rho_1 + p_1), \tag{34}$$

$$\dot{\rho_2} = -2\dot{\lambda}(\rho_2 + p_2), \tag{35}$$

$$-\dot{p_1} + 2p_1 = \dot{\nu}(\rho_1 + p_1), \tag{36}$$

$$-\dot{p_2} = \dot{\nu}(\rho_2 + p_2), \tag{37}$$

$$\rho_1 = -4\dot{\lambda} - 3\dot{\lambda}^2 - 2\ddot{\lambda} - 1, \tag{38}$$

$$e^{2\nu}\rho_2 = \dot{\lambda}^2, \tag{39}$$

$$0 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\lambda}\dot{\nu}, \tag{40}$$

$$p_1 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\nu} + 2\dot{\lambda}\dot{\nu}, \tag{41}$$

$$e^{2\nu}p_2 = 2\dot{\lambda}\dot{\nu} - 2\ddot{\lambda} - 3\dot{\lambda}^2, \tag{42}$$

$$p_1 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda}\dot{\nu} + \ddot{\nu} + \dot{\nu}^2, \tag{43}$$

$$e^{2\nu}p_2 = -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\lambda}\dot{\nu}. \tag{44}$$

For the EOS(3) when $k \neq -1$, $\rho_2 = 0$, and ρ_1 is arbitrary, Eq. (39) yields $\dot{\lambda} = 0$ while Eqs. (34) and (36) show that $\dot{\nu} = \frac{2k}{k+1}$. Finally, we obtain the same solution as in the case of the second kind with the EOS(3) given by Eq. (30) with $\alpha = 0$. The corresponding metric is

$$ds^2 = (xe^{-t})^{\frac{4k}{k+1}} dt^2 - dx^2 - e^{2t}(dy^2 + dz^2). \tag{45}$$

For the self-similarity of the first kind in the orthogonal perfect fluid case, the EFEs and the equations of motion give

$$
\begin{aligned}
e^{2\nu}(1+\rho) &= \lambda'^2, && (46)\\
e^{2\nu}(3-p) &= 3\lambda'^2 + 2\lambda'' - 2\lambda'\nu', && (47)\\
e^{2\nu}(1-p) &= \lambda'' + \lambda'^2 - \lambda'\nu', && (48)\\
2\lambda'(\rho+p) &= -\rho', && (49)\\
\rho &= p. && (50)
\end{aligned}
$$

Clearly, Eq. (50) shows that this is a system with a stiff fluid. If these equations are solved simultaneously, Eq. (49) provides the value of λ', and Eq. (46) gives the value of ν in terms of p. Equations (47) and (48) impose a constraint on p, $p'^2 p - 2(1+p)(p''p - p'^2) = 0$, and we arrive at the following solution:

$$
\nu = \ln\left(\frac{p'}{4p\sqrt{(1+p)}}\right), \quad \lambda = -\frac{1}{4}\ln(p) + \ln(c_1), \quad \rho = p. \qquad (51)
$$

For the self-similarity of the second kind in the orthogonal perfect fluid case, the quantities ρ and p must be of the forms

$$
\begin{aligned}
\kappa\rho &= x^{-2}\rho_1(\xi) + x^{-2\alpha}\rho_2(\xi), && (52)\\
\kappa p &= x^{-2}p_1(\xi) + x^{-2\alpha}p_2(\xi). && (53)
\end{aligned}
$$

A set of ODEs gives

$$
\begin{aligned}
\rho_1' &= -2\lambda'(\rho_1 + p_1), && (54)\\
\rho_2' &= -2\lambda'(\rho_2 + p_2), && (55)\\
2p_1 &= \alpha(\rho_1 + p_1), && (56)\\
\rho_2 &= p_2 && (57)\\
\rho_1 &= -1, && (58)\\
e^{2\nu}\rho_2 &= \lambda'^2, && (59)\\
0 &= (1-\alpha)\lambda', && (60)\\
p_1 &= 1 + 2\alpha, && (61)\\
e^{2\nu}p_2 &= -2\lambda'' + 2\lambda'\nu' - 3\lambda'^2, && (62)\\
p_1 &= \alpha^2, && (63)\\
e^{2\nu}p_2 &= -\lambda'' - \lambda'^2 + \lambda'\nu'. && (64)
\end{aligned}
$$

Equation (57) represents a stiff fluid, and Eq. (60) gives $\lambda' = 0$; hence, we obtain the following solution:

$$\begin{aligned} \nu &= arbitrary, \quad \lambda = c_4, \quad p_2 = 0 = \rho_2, \quad \rho_1 = -1, \\ p_1 &= 3 \pm 2\sqrt{2}, \quad \alpha = 1 \pm \sqrt{2}. \end{aligned} \tag{65}$$

The corresponding metric is

$$ds^2 = x^{2(1\pm\sqrt{2})} dt^2 - dx^2 - x^2(dy^2 + dz^2). \tag{66}$$

For the self-similarity of the infinite kind in the parallel perfect fluid, a set of ODEs is given as

$$-\rho = 3\lambda'^2 + 2\lambda'', \tag{67}$$

$$p = \lambda'^2 + 2\lambda'\nu', \tag{68}$$

$$p = \lambda'' + \lambda'^2 + \lambda'\nu' + \nu'' + \nu'^2, \tag{69}$$

$$-p' = \nu'(\rho + p). \tag{70}$$

Solving Eqs. (67)-(70) with the assumption that p is constant, we find that λ is a linear function of ξ. Finally, we arrive at the following solution:

$$\begin{aligned} \nu &= c_1, \quad \lambda = c_2\xi + c_3, \\ \rho &= -3p = constant. \end{aligned} \tag{71}$$

The metric is given by

$$ds^2 = dt^2 - dx^2 - e^{2x}(dy^2 + dz^2). \tag{72}$$

We notice that the solutions given by Eqs.(14), (31) and (66) turn out to be dependent and the solutions given by Eqs. (29), (45), (51), and (72) are independent. Thus, we have a total of five independent solutions. It is worth mentioning here that the self-similar solutions in Eqs. (14), (31), and (66) correspond to the already classified solutions [3] under particular coordinate transformations. The metrics given by Eqs. (14), (31), and (66) correspond to the class of metrics

$$ds^2 = e^{2\nu(x)} dt^2 - dx^2 - e^{2\lambda(x)}(dy^2 + dz^2), \tag{73}$$

which has four Killing vectors admitting $G_3 \otimes \Re$ with a timelike $\Re$. We also note that the density is either zero or positive in all the solutions, except for

the solution given by Eq. (66). The physical properties of all these solutions can be seen in the Ref. 4.

We would like to mention here that the paper in Ref. 1 focussed on a classification of plane symmetric kinematic self-similar solutions under certain assumptions. A complete classification for the most general plane symmetric kinematic self-similar solutions appears elsewhere [2].

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